Nonlinear systems

Nonlinearity is the rule not the exception.

A function is linear if $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)$ for every $x, y \in D_f$ and all scalars α and β.

$\beta = 0 \quad \Rightarrow \quad f(\alpha x) = \alpha x$ Proportionality represents a linear sytem.
$\alpha, \beta = 0 \quad \Rightarrow \quad f(0) = 0$ $f(x) = ax + b$ is linear only if $b = 0$ but loosely we think of it as linear.

$f(z) = z^2$ is a properly non-linear function since $(x + y)^2 \neq x^2 + y^2$.

The concept of linearity can be widened to operators acting on functions.
An operator $\mathcal{L}$ is linear if $\mathcal{L}(\alpha \cdot f + \beta \cdot g) = \alpha \cdot \mathcal{L}(f) + \beta \cdot \mathcal{L}(g)$ for all functions f and g.

An equation $f(x) = 0$ or functional equation $\mathcal{L}(f) = 0$ is linear if the function f or operator $\mathcal{L}$ is linear.

Ex: $\mathcal{L}(f) = 3x \cdot \dfrac{d^2 f}{dx^2} + 5\dfrac{df}{dx}$ is linear in $f \quad \Longrightarrow \quad 3xy'' + 5y' = 0$ is a linear differential equation

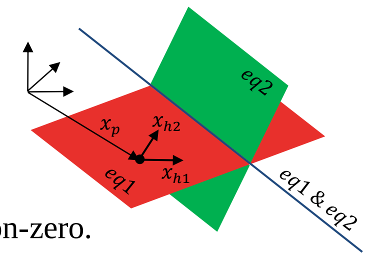
The linear property divides into two parts, additivity and homogeneity:

- Additivity: $f(x + y) = f(x) + f(y)$
- Homogeneity: $f(\alpha x) = \alpha f(x)$

Additivity means that two solutions add up to a third solution:
$$f(x) = 0\,, f(y) = 0 \Rightarrow f(x + y) = 0$$
$$\mathcal{L}(f) = 0\,, \mathcal{L}(g) = 0 \Rightarrow \mathcal{L}(f + g) = 0$$

Homogeneity means that one solution gives a whole ray of solutions:
$$f(x) = 0 \;\Rightarrow f(cx) = 0$$
$$\mathcal{L}(f) = 0 \;\Rightarrow\; \mathcal{L}(cf) = 0$$

All the solutions of a linear problem add up to a linear subspace,
spanned by some set of basic solutions, a base.
All this is basic stuff of linear algebra and the study of vector spaces.



Equations with non-zero r.h.s. $f(x) = k$ and $\mathcal{L}(g) = h$ with linear f and $\mathcal{L}$
are called homogenous if k and g are zero and inhomogeneous if they are non-zero.

The general solution becomes $x = x_p + x_h$ and $g = g_p + g_h$ where index p belongs to one particular solution and index h is part of the general solution to the homogenous equations $f(x_h) = 0$ and $\mathcal{L}(g_h) = 0$.

Additivity $f(x + y) = f(x) + f(y)$ for a real function $f\colon \mathbb{R} \to \mathbb{R}$ implies
homogeneity $f(\alpha x) = \alpha f(x)$ for every $\alpha \in \mathbb{Q}$ and for all $\alpha \in \mathbb{R}$ if f is continuous.

Exercise: Show that any additive function $f\colon \mathbb{Q} \to \mathbb{Q}$ is of the form $f(q) = cq$ for some constant $c \in \mathbb{Q}$.

Looking for solutions to the functional equation $f(x + y) = f(x) + f(y)$ [Cauchy's functional equation]
among the non-continuous functions leads to some weird looking functions, very different from $f(x) = cx$.
These strange functions can be derived by using the axiom of choice and non-constructive methods.
A sign of their pathological nature is that they are dense in $\mathbb{R}^2$,
every disk in the plane no matter how small contains a point from the graph $\big(x, f(x)\big)$.

Systems that are based on linearity like $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ are well behaved, and a bit dull.
Maybe not that dull, after all Maxwell's equations represent a linear system of differential equations.

A system that is not linear is called nonlinear, it opens up for all kinds of interesting things.